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Symmetry properties of a system of Euler-type equations for magnetized plasmas

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Abstract
We provide all Lie point symmetries of a system of partial differential equations, of great interest to plasma physics; we deduce the corresponding invariant solutions and discuss their physical interpretation. We also consider some of its conditional and partial symmetries and obtain other interesting solutions. Finally, we point out that, although the system can be cast in divergence form and admits conserved currents, it does not admit potential symmetries.

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1. Introduction

It is well known that the analysis of symmetry properties is one of the most interesting tools in the study of both ordinary and partial differential equations, and of systems thereof (e.g., [1–7] and references therein); it is also known that this analysis can be of concrete help in finding explicit solutions to these equations.

At present, finding symmetries of a given differential equation is an (almost) completely standard routine, thanks also to some computer packages (see, e.g., [8, 9]) which can help in the calculations (we will consider only Lie point-symmetries, see below). While this is true for what concerns *exact* symmetries, this may not be completely obvious when weaker and more refined notions of symmetry, such as conditional, partial or potential symmetry, are considered.

Although these notions are not new, the system of partial differential equations that we analyse here (which is of considerable interest to plasma physics) is so rich in various symmetry properties and admits so different types of nontrivial symmetries and particular solutions, that we believe that this application of symmetry methods deserves to be proposed to both specialists in plasma physics and experts in symmetry theory.

Indeed, the interest of this paper may lie in different features for different types of readers. For those working in plasma physics the interest may be in the obtained solutions, which, to the best of our knowledge, are new and exhibit physically relevant properties, which will be discussed in more detail in a future paper. For readers more interested in theoretical aspects of symmetry methods applied to PDEs, it can be of interest to see new 'real world' applications and not just illustrative examples.

Our system of equations has been widely used recently in the investigation of 'magnetic field line reconnection' (see [10–13]) and of the interaction of 'magnetized plasma vortices' (see [14–16]). In addition, the algebraic structure of these coupled partial differential equations is of interest, as each of them has the same algebraic structure of the incompressible two-dimensional Euler equation written in terms of the fluid vorticity. For other applications of symmetry analysis to other relevant equations in plasma physics, see, e.g., [17–19].

In order to fix notation, and also possibly to propose the paper to a larger audience, let us briefly recall the main procedure (see, e.g., [2] for details). Given a system $\Delta \equiv \Delta_\ell(x, u^{(m)}) = 0$ of ℓ partial differential equations for the q-dependent variables (or unknown functions) $u \equiv u_a(x)$ $(a = 1, \ldots, q)$ of the p-independent variables $x \equiv (x_1, \ldots, x_p)$ ($u^{(m)}$ denotes the derivatives of u up to some order m), a vector field X,

$$X = \xi_i(x, u)\frac{\partial}{\partial x_i} + \zeta_a(x, u)\frac{\partial}{\partial u_a} \tag{1}$$

(sum over repeated indices) is the Lie generator of an (exact) symmetry for the system $\Delta = 0$ if the appropriate prolongation X^* of X satisfies

$$X^*(\Delta)|_{\Delta=0} = 0, \tag{2}$$

i.e., if $X^*(\Delta)$ vanishes along the solutions of $\Delta = 0$. If X is a symmetry, then (once any solution to $\Delta = 0$ is given) one may obtain, under the action of X, an orbit of solutions to $\Delta = 0$. There may exist solutions $u^0 = u^0(x)$ which are left fixed under X: these satisfy the invariance condition

$$X_Q u^0 = 0 \tag{3}$$

where

$$X_Q = \left(\zeta_a - \xi_i\frac{\partial u_a}{\partial x_i}\right)\frac{\partial}{\partial u_a} \tag{4}$$

is the vector field X written in 'evolutionary form' [2].

In addition to these (exact) symmetries, we will also consider conditional and partial symmetries (we shall refer here to the definition of partial symmetry as given in [20]); these notions will be recalled in sections 3 and 4, respectively. In section 5 we will also briefly deal with the case of potential symmetries.

2. Exact symmetries

The system $\Delta = 0$ of partial differential equations that we consider is as follows:

$$\begin{cases} \dfrac{\partial}{\partial t}(\psi - \nabla^2\psi + \nabla^2\varphi) + [\varphi + \psi, \psi - \nabla^2\psi + \nabla^2\varphi] = 0 \\ \dfrac{\partial}{\partial t}(\psi - \nabla^2\psi - \nabla^2\varphi) + [\varphi - \psi, \psi - \nabla^2\psi - \nabla^2\varphi] = 0. \end{cases} \tag{5}$$

Here $x \equiv (x, y, t)$, $u \equiv (\psi, \varphi)$, $\psi = \psi(x, y, t)$, $\varphi = \varphi(x, y, t)$, $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ and

$$[f, g] = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial g}{\partial x}\frac{\partial f}{\partial y}.$$

The above equations describe the low-frequency nonlinear evolution of a two-dimensional plasma configuration embedded in a strong magnetic field in the z-direction and with a shear magnetic field in the x–y plane $\mathbf{B} = B_0\mathbf{e}_z + \nabla\psi \times \mathbf{e}_z$. As detailed in the appendix, these equations can be obtained from a generalized two-fluid dissipationless model of the plasma response where the plasma velocity in the x–y plane is given by the electric drift $\mathbf{v} = \mathbf{e}_z \times \nabla\varphi$, φ is proportional to the electric potential in the plasma and plays the role of a stream function, while ψ is a magnetic flux function proportional to the z component of the magnetic vector potential [10–13]. This model includes the effects of the electron inertia and of the electron pressure. For the sake of simplicity, here we have adopted a suitable rescaling of the variables and set all physical dimensionless parameters equal to 1; see also section 4.3. The boundary conditions that this system of equations obeys is briefly discussed at the end of the appendix.

There are some obvious symmetries of the above system, namely spatial and time translations, and spatial rotations, generated respectively by

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial y}, \qquad X_3 = \frac{\partial}{\partial t}, \qquad X_4 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}.$$

Other trivial symmetries are given by the vector fields

$$X_5 = \frac{\partial}{\partial \psi}, \qquad X_H = H(t)\frac{\partial}{\partial \varphi}$$

which generate the transformations $\psi \to \psi + k$, $\varphi \to \varphi + H(t)$, where k is a constant and $H(t)$ an arbitrary function, which do not change either the magnetic field or the plasma velocity.

With the help of some appropriate computer package, e.g. [8, 9], it is possible to show the following result:

Proposition 1. *Apart from trivial symmetries, the system (5) admits only the following symmetries: the infinite-dimensional subalgebra*

$$X_{(A,B)} = A(t)\frac{\partial}{\partial x} + B(t)\frac{\partial}{\partial y} + \left(x\frac{\mathrm{d}B}{\mathrm{d}t} - y\frac{\mathrm{d}A}{\mathrm{d}t}\right)\frac{\partial}{\partial \varphi} \tag{6}$$

where $A(t)$, $B(t)$ are (nonconstant) arbitrary differentiable functions, and the vector field

$$X_6 = -ty\frac{\partial}{\partial x} + tx\frac{\partial}{\partial y} + \frac{x^2 + y^2}{2}\frac{\partial}{\partial \varphi}. \tag{7}$$

The nonzero (and nontrivial) commutation rules between the above symmetry generators are as follows:

$$\begin{aligned} &[X_1, X_{(A,B)}] = X_{H_1}, && [X_1, X_6] = X_{(0,t)} \\ &[X_3, X_6] = -X_4, && [X_4, X_{(A,B)}] = X_{(-B,A)} \\ &[X_{(A,B)}, X_{(C,D)}] = X_{H_2}, && [X_3, X_{(A,B)}] = X_{(A_t,B_t)} \\ &[X_{(A,B)}, X_6] = X_{(-tB,tA)} \end{aligned}$$

where $A_t = \mathrm{d}A/\mathrm{d}t$ (similar for $B(t)$, $C(t)$ and $D(t)$), $H_1 = B_t$ and $H_2 = AC_t - BD_t - CB_t + DA_t$. These commutation rules confirm that the obtained algebra is closed.

We will now consider in some detail the two above symmetries (6) and (7).

(1) The subalgebra $X_{(A,B)}$ describes an infinite family of symmetries, due to the presence of the functions $A(t)$, $B(t)$. This expresses the property that, if $\psi(x, y, t)$, $\varphi(x, y, t)$ is a solution of (5), then

$$\begin{aligned} &\Psi(x, y, t) := \psi(x - A(t), y - B(t), t), \\ &\Phi(x, y, t) := xB_t - yA_t - \tfrac{1}{2}(AB_t - A_tB) + \varphi(x - A(t), y - B(t), t) \end{aligned}$$

also[3] solve our system (5), for any $A(t)$, $B(t)$. It must be noted that this corresponds to a change of spatial coordinates into a moving frame which produces in turn the additional term $xB_t - yA_t - (1/2)(AB_t - A_tB)$ in the component φ. Since φ is proportional to the electric potential, we can interpret this symmetry as expressing the fact that a time-dependent, spatially uniform, electric field imposed on the system induces a uniform time-dependent electric drift $A_t\mathbf{e}_x + B_t\mathbf{e}_y$.

We now look for the solutions to (5) which are *invariant* under (6), i.e. for solutions to (5) satisfying the invariance condition (3), which now takes the form ($\psi_x = \partial\psi/\partial x$, etc)

$$A\psi_x + B\psi_y = 0, \qquad A\varphi_x + B\varphi_y = xB_t - yA_t.$$

The first equation can be interpreted as the requirement that the displacement $A\mathbf{e}_x + B\mathbf{e}_y$ produced by the electric drift $A_t\mathbf{e}_x + B_t\mathbf{e}_y$ be parallel to the field lines of the shear magnetic field $\nabla\psi \times \mathbf{e}_z$, while the second expresses the requirement that the electric potential of the displaced plasma element remain constant.

The above equations may easily be integrated to get

$$\psi = V(s,t), \qquad \varphi = \frac{1/2}{A^2+B^2}((A_tB + AB_t)(x^2 - y^2) - 2xy(AA_t - BB_t)) + W(s,t) \tag{8}$$

where $s = B(t)x - A(t)y$ and t are $X_{A,B}$-invariant variables, so that ∇s is orthogonal to the plasma displacement; the configuration (8) corresponds to a magnetic field aligned along the direction of the displacement and to a time-dependent velocity pattern consisting of a hyperbolic field with stagnation point at $x = y = 0$ superimposed to a field uniform along the plasma displacement.

Substituting (8) into the system (5) we obtain the following 'reduced' system for the functions $V = V(s,t)$, $W = W(s,t)$,

$$\frac{\partial}{\partial t}(V - (A^2 + B^2)V_{ss}) = 0, \qquad \frac{\partial}{\partial t}((A^2 + B^2)W_{ss}) = 0$$

which clearly imply

$$V - (A^2 + B^2)V_{ss} = F(s), \qquad (A^2 + B^2)W_{ss} = G(s)$$

where $F(s)$, $G(s)$ are arbitrary functions. Note that $\partial_{xx} + \partial_{yy} = (A^2 + B^2)\partial_{ss}$ and that these equations are actually ODEs; indeed, the variable t here appears merely as a parameter.

Special simple solutions with a uniform magnetic field and a purely hyperbolic velocity field can be obtained with $V(s) = s$, i.e. $\psi = B(t)x - A(t)y$, and $W = 0$ in (8). In the elementary case $A(t) = \pm B(t) = A_0 \exp(t)$ this solution corresponds to the exponential growth of the magnetic field amplitude in a time-independent velocity field, while in the case $A(t) = A_0\cos(\omega t)$, $B(t) = A_0\sin(\omega t)$ it corresponds to a magnetic field rotating with frequency ω in a velocity field rotating with frequency 2ω.

It is remarkable that this reduced system consists of two *linear and uncoupled* homogeneous equations; in particular, it produces solutions where ψ satisfies a *linear* equation (therefore linear superposition principle holds), and φ does the same, apart from a fixed additional term. The fact that the invariant solutions describe a linear manifold may appear rather surprising: we shall comment on this point at the end of this section.

(2) The other symmetry X_6 (equation (7)) does not depend on arbitrary functions; it implies that if $\psi(x,y,t)$, $\varphi(x,y,t)$ is a solution of (5), then also

$$\Psi(x,y,t) := \psi(x\cos(\lambda t) + y\sin(\lambda t), -x\sin(\lambda t) + y\cos(\lambda t), t)$$

[3] The Lie parameter λ which should be introduced to parametrize this family of solutions can be clearly absorbed in the (arbitrary) functions A and B.

$$\Phi(x, y, t) := \varphi(x\cos(\lambda t) + y\sin(\lambda t), -x\sin(\lambda t) + y\cos(\lambda t), t) + \lambda\frac{x^2+y^2}{2}$$

are a family of solutions to (5) for any $\lambda \in \mathbf{R}$. This represents a sort of rotated solutions with angular velocity λ plus a radial term in the component φ which gives the additional velocity field corresponding to the rotation. As a trivial example, starting from the simple solution $\psi = \exp(-x^2)$, $\varphi = 1/(1+x^2)$ of (5), we can conclude that

$$\Psi(x, y, t) := \exp(-(x\cos(\lambda t) + y\sin(\lambda t))^2)$$
$$\Phi(x, y, t) := (1 + (x\cos(\lambda t) + y\sin(\lambda t))^2)^{-1} + \lambda\frac{x^2+y^2}{2}$$

also solve (5).

We now look for solutions to (5) which are *invariant* under (7), i.e. for solutions to (5) satisfying the invariance condition (3), which now reads

$$y\frac{\partial\psi}{\partial x} - x\frac{\partial\psi}{\partial y} = 0, \qquad y\frac{\partial\varphi}{\partial x} - x\frac{\partial\varphi}{\partial y} = -\frac{r^2}{2t}$$

where $r^2 = x^2 + y^2$. It is easy to find that these X_2-invariant solutions to (5) are of the form, with $\theta = \arccos(x/r)$,

$$\psi = Q(r, t), \qquad \varphi = \frac{r^2}{2t}\theta + R(r, t).$$

As in the above case, substituting in (5), one obtains that the functions Q and R must satisfy two *linear* and uncoupled homogeneous equations:

$$r^2 Q_{rrr} - 2rtQ_{rrt} + rQ_{rr} - 2tQ_{rt} - r^2 Q_r + 2rtQ_t + 3Q_r = 0$$
$$2rtR_{rrt} - r^2 R_{rrr} + 2tR_{rt} - rR_{rr} + 5R_r = 0.$$

It can be noted in particular the special form of the component φ, which has a fixed term containing a 'cut' discontinuity, which looks like a spiral and expresses the fact that this solution contains a θ-independent azimuthal electric field, which vanishes for $t \to \pm\infty$; whereas the equation for the other term $R(r, t)$ admits solutions of the form

$$R = t^a r^b \qquad \text{with} \quad a = \frac{b^2 - 2b - 4}{2b}, \quad \forall b.$$

We can now show why all the symmetries considered above yield invariant solutions which belong to a linear manifold. This follows from this simple result.

Lemma 1. *Assume that the vector field X has the following specific form,*

$$X = \xi(x, y, t)\frac{\partial}{\partial x} + \eta(x, y, t)\frac{\partial}{\partial y} + \tau(x, y, t)\frac{\partial}{\partial t} + \zeta(x, y, t, \psi)\frac{\partial}{\partial\psi} + \chi(x, y, t, \varphi)\frac{\partial}{\partial\varphi} \tag{9}$$

(i.e., in particular, X is a projectable), then the general X-invariant solutions to (5) take the form

$$\psi = \alpha_1(x, y, t) + Z_1(s_1, s_2), \qquad \varphi = \alpha_2(x, y, t) + Z_2(s_1, s_2) \tag{10}$$

where α_1, α_2 are fixed functions, s_1, s_2 are X-invariant coordinates and Z_1, Z_2 are solutions of the reduced system. If in addition $[s_i, s_j] = 0$, then Z_1, Z_2 satisfy linear equations, and the invariant solutions have the form (10) of a fixed function α_i plus a term Z_i belonging to a linear space.

Proof. The proof comes from direct calculation. The general invariant solutions under X can be found solving the characteristic equation

$$\frac{dx}{\xi} = \frac{dy}{\eta} = \frac{dt}{\tau} = \frac{d\psi}{\zeta} = \frac{d\varphi}{\chi}. \tag{11}$$

Now, the invariant coordinates s_1, s_2 are obtained from the subsystem $\mathrm{d}x/\xi = \mathrm{d}y/\eta = \mathrm{d}t/\tau$, the functions α_1, α_2 are determined by the remaining equations in (11), and finally Z_1, Z_2 must satisfy the reduced equations obtained substituting (10) into (5). It is now easy to see that all nonlinear terms in the equations containing Z_i $(i = 1, 2)$ come from $[Z_i, Z_j]$ and $[Z_i, \nabla^2 Z_j]$ in (5), and a direct substitution shows that these quantities are proportional just to $[s_i, s_j]$; therefore they disappear if $[s_i, s_j] = 0$. □

All assumptions of the above lemma are satisfied in the cases considered above; let us remark in particular that the condition $[s_i, s_j] = 0$ is automatically satisfied if, e.g., one of the invariant variables s_i can be chosen to depend only on the time t.

As seen in the above discussion, there are mainly two applications of symmetry properties of a differential problem, i.e. finding new solutions starting from a known one, and looking for invariant solutions under the symmetry. The introduction of the conditional and partial symmetries, which are *not* exact symmetries of the problem, has essentially the scope of preserving just one of these two features of exact symmetries.

3. Conditional symmetries

We will consider first the case of conditional symmetries [21–25], although in the case of our problem (5) more interesting situations occur in the presence of partial symmetries (considered in the next section).

As is well known, a conditional symmetry for an equation (or system of equations) $\Delta = 0$ is, roughly, a vector field X which is a symmetry of the enlarged system obtained appending to the initial equation $\Delta = 0$ the equation expressing the invariance under X, i.e. $X_Q u = 0$, as in (3). It must also be required that this system does admit some solution. It is clear that if X is a conditional symmetry (and not an exact symmetry), there are no (proper) orbits of solutions, indeed conditional symmetries do not map, in general, solutions of $\Delta = 0$ into other solutions.

There are some delicate points related to this definition; see [26–30]. In particular, it has been pointed out [26] that, given an equation $\Delta = 0$, *any* vector field X can be a conditional symmetry, the only condition to be verified is the existence of some solutions to the system $\Delta = 0, X_Q u = 0$. Conversely, one can say that *any* solution to $\Delta = 0$ can be considered an invariant solution under some vector field X. This remark is actually related to the introduction of a subtler classification of the notion of conditional symmetries [31], which will not be considered here, and which depends on the number of auxiliary equations (differential consequences) needed for solving the problem. Therefore, if one is not interested in this classification, but only in finding solutions of the problem $\Delta = 0$, one just has to choose 'reasonable' (and physically relevant, of course) vector field s X and check if the system $\Delta = X_Q u = 0$ admits solutions; these solutions, by construction, are invariant under X.

It often happens that the solutions found in this way are trivial (e.g., $u = \text{const}$), or may be obtained by means of different procedures. For example, considering for our system (5) the case of spatial dilations (scalings)

$$X = ax\frac{\partial}{\partial x} + by\frac{\partial}{\partial y} \qquad a, b = \text{const}$$

which are not exact symmetries, we get the disappointing result that the only solution to (5) and $X_Q u = 0$ must be independent of the time, $\psi_t = \varphi_t = 0$, and must satisfy $\psi = \pm\varphi$. Unfortunately, this does not produce new interesting solutions, because *any* couple of functions $\psi_0(x, y)$ and $\varphi_0(x, y) = \pm\psi_0(x, y)$ are solutions to $\Delta = 0$, although not scaling invariant!

More useful indications are given by the vector field

$$X = \psi_y \frac{\partial}{\partial \psi} + \varphi_x \frac{\partial}{\partial \varphi} \tag{12}$$

which is actually a vector field in evolutionary form not reducible to a vector field in standard form (1) (or a 'contact symmetry' [4]). The X-invariant solutions of (5) which can be found are as follows,

$$\psi = \sin[k(x - \gamma(t))], \qquad \varphi = \exp[\pm(1 + k^2)^{1/2} y] - \frac{\mathrm{d}\gamma}{\mathrm{d}t} y + T(t)$$

or

$$\psi = \exp[\pm(1 + k^2)^{1/2}(x - \gamma(t))], \qquad \varphi = \sin(ky) - \frac{\mathrm{d}\gamma}{\mathrm{d}t} y + T(t)$$

where γ and T are arbitrary functions and $k \in \mathbf{R}$, or also, with $|\kappa| < 1$,

$$\psi = \exp[\pm(1 - \kappa^2)^{1/2}(x - \gamma(t)], \qquad \varphi = \exp(\pm\kappa y) - \frac{\mathrm{d}\gamma}{\mathrm{d}t} y + T(t).$$

Note, in particular, the presence of terms depending on $x - \gamma(t)$ describing a generalized wave propagation. Other solutions which can be obtained in the same way are

$$\psi = \gamma(t)\exp(\pm x), \qquad \varphi = ky^2 + yT_1(t) + T(t)$$

(with γ, T_1 and T arbitrary functions), and

$$\psi = \gamma(t) + kx, \qquad \varphi = \exp(\pm y) + \frac{\mathrm{d}\gamma}{\mathrm{d}t}\frac{y}{k} \qquad (k \neq 0)$$

and similar solutions of the same form, obtained changing sin into cos, adding constant terms and so on.

Needless to say, starting from these solutions and using the exact symmetries of (5) examined in section 2, one may construct other families of solutions to (5).

4. Partial symmetries

We will refer to the notion of partial symmetry which has been introduced in [20] (the term 'partial symmetry' has been used with different meanings in the past literature [1, 32]; see also [33]). While exact symmetries transform any solution of the given problem into another solution, partial symmetries do the same only for a proper subset of solutions, which is defined by some supplementary differential equations. More precisely, let us assume that X is *not* an exact symmetry, therefore $X^*(\Delta)|_{\Delta=0} \neq 0$; let us then introduce the condition

$$\Delta^{(1)} := X^*(\Delta) = 0 \tag{13}$$

as a new equation, and consider the enlarged system

$$\Delta = \Delta^{(1)} = 0. \tag{14}$$

It is clear that if X is an exact symmetry of this system, then the subset $\mathcal{S}^{(1)}$ of the simultaneous solutions of this system is a 'symmetric set of solutions' to $\Delta = 0$, i.e. a proper subset of solutions which have the property of being mapped from the one into another by the vector field X. It is also clear that this property is not shared by the other solutions to $\Delta = 0$ not belonging to the subset $\mathcal{S}^{(1)}$. In principle, this procedure can be iterated introducing, in the case X is not a symmetry of the enlarged system (14), further equations $\Delta^{(2)} := X^*(\Delta^{(1)}) = 0$ and so on (examples of this procedure can be found in [20, 33]), but for the examples we consider in this paper, no further iteration after the first step (13), (14) is needed.

4.1. Example 1

As a first example of partial symmetry for the problem (5), consider the vector field of the form

$$X_{ab} = -ty\frac{\partial}{\partial x} + tx\frac{\partial}{\partial y} + \frac{x^2+y^2}{2}\left(a\frac{\partial}{\partial \psi} + b\frac{\partial}{\partial \varphi}\right) \tag{15}$$

where a, b are constants (not both zero), which is similar to (7) but now also involves the component ψ. According to our procedure, let us evaluate $X^*_{ab}(\Delta)$: we obtain a system of equations which can be rewritten in this form,

$$\Delta^{(1)} := X^*_{ab}\Delta = \begin{cases} \left(y\dfrac{\partial}{\partial_x} - x\dfrac{\partial}{\partial y}\right)((1-b)(\psi - \nabla^2\psi) + a(\varphi - \nabla^2\varphi)) = 0 \\ \left(y\dfrac{\partial}{\partial_x} - x\dfrac{\partial}{\partial y}\right)(a\nabla^2\psi + (1-b)\nabla^2\varphi) = 0. \end{cases} \tag{16}$$

It is clear that $X^*_{ab}(\Delta) = 0$ if and only if $a = 0, b = 1$, which corresponds to the exact symmetry (7) already considered in section 2. Excluding this case, it is not difficult to show that the enlarged system (14), which is now given by the four equations (5) and (16), admits X_{ab} as an exact symmetry, i.e. $X^*_{ab}(\Delta) = X^*_{ab}(\Delta^{(1)}) = 0$ when $\Delta = \Delta^{(1)} = 0$. Therefore, X_{ab} is a partial symmetry, and the set $\mathcal{S}^{(1)}$ of the simultaneous solutions of the system (5), (16) (which is clearly a subset of all the solutions of $\Delta = 0$, i.e. of (5)), is symmetric under X_{ab}, and as a consequence, the solutions belonging to this set are transformed by X_{ab} into other solutions to (5). Written explicitly, if $\psi(x, y, t), \varphi(x, y, t)$ solve (5) and (16), then

$$\Psi(x, y, t) := \psi(x\cos(\lambda t) + y\sin(\lambda t), -x\sin(\lambda t) + y\cos(\lambda t), t) + a\lambda\frac{x^2+y^2}{2}$$

$$\Phi(x, y, t) := \varphi(x\cos(\lambda t) + y\sin(\lambda t), -x\sin(\lambda t) + y\cos(\lambda t), t) + b\lambda\frac{x^2+y^2}{2}$$

also solve (5), for all $\lambda \in \mathbf{R}$.

Different choices of the constants a, b give different interesting possibilities:

(i) If $a = 0, b \neq 0, 1$, then the additional equation $\Delta^{(1)} = 0$ is satisfied by functions ψ, φ such that

$$\psi - \nabla^2\psi = F_1(r, t), \qquad \nabla^2\varphi = F_2(r, t) \tag{17}$$

where (here and in the remainder of this subsection) $F_i(r, t)$ denote arbitrary functions of $r = \sqrt{x^2+y^2}$ and the time.

(ii) if $a \neq 0, b = 1$, then the additional equation is satisfied if

$$\nabla^2\psi = F_3(r, t), \qquad \varphi - \nabla^2\varphi = F_4(r, t).$$

(iii) if $a \neq 0, b = 1 \pm a$, then the additional equation is satisfied if

$$\psi \pm \varphi = F_5(r, t).$$

We give just a simple example for case (ii), with $b = 2$: observing, e.g., that $\psi = 3x^2 + y^2, \varphi = \exp(-x)$ satisfy both (5) and (17), we deduce that also

$$\Psi = 3(x\cos(\lambda t) + y\sin(\lambda t))^2 + (-x\sin(\lambda t) + y\cos(\lambda t))^2 + \lambda\frac{x^2+y^2}{2},$$

$$\Phi = \exp(-(x\cos(\lambda t) + y\sin(\lambda t))) + \lambda(x^2+y^2)$$

is a family of solutions of (5), $\forall\,\lambda \in R$.

Direct verification can also show that for no choice of a, b (apart from the cases $a = b = 0$ and $a = 0, b = 1$ corresponding to exact symmetries) there are invariant solutions under X_{ab}; this means that the vector field X_{ab} is *not* a conditional symmetry for (5).

4.2. Example 2

To show the usefulness of the notion of partial symmetry, let us consider the vector field

$$X = \psi \frac{\partial}{\partial \psi}. \tag{18}$$

First of all, note that this is trivially a conditional symmetry for (5), indeed the invariance condition $X_Q u = 0$ is now $\psi = 0$ and therefore simply amounts to looking for the special solutions to (5) with $\psi = 0$, i.e to the hydrodynamic limit. It can be more interesting to show that the vector field (18) is a nontrivial partial symmetry: indeed, considering this vector field corresponds to looking for solutions to (5) such that the component ψ admits a scaling property, i.e., for solutions ψ, φ such that $\lambda\psi, \varphi$ also solve (5) for all $\lambda \in \mathbf{R}$. Applying the prolongation X^* to the system (5), and combining the resulting equation $\Delta^{(1)} = X^*(\Delta)$ with (5), one gets the new condition

$$[\psi, \nabla^2\psi] = 0 \tag{19}$$

which is then the condition characterizing the subset $\mathcal{S}^{(1)}$ of solutions with the above specified property. It is easy to verify that the system of the three equations (5) and (19) is symmetric under (18), showing that (18) is indeed a partial symmetry for (5). Using (19) one can also rewrite the system (5), (19) in the more convenient form

$$\begin{aligned} &\frac{\partial}{\partial t}(\psi - \nabla^2\psi) + [\varphi, \psi - \nabla^2\psi] + [\psi, \nabla^2\varphi] = 0 \\ &\frac{\partial}{\partial t}(\nabla^2\varphi) + [\varphi, \nabla^2\varphi] = 0 \\ &[\psi, \nabla^2\psi] = 0. \end{aligned} \tag{20}$$

The interesting feature of this system is that the (second) equation in (20) for φ is independent of ψ and is equivalent to the two-dimensional Euler equation for an incompressible fluid, and that, once φ is given, the first equation for ψ is linear: obviously, this agrees with the presence of the (partial) symmetry given by (18).

Let us remark that condition (19) implies that $\nabla^2\psi = A(\psi, t)$ for some smooth function A, therefore the first equation in (20) can be rewritten in the equivalent form ($A_\psi = \partial A/\partial\psi$, etc):

$$(1 - A_\psi)(\psi_t - [\psi, \varphi]) = A_t - [\psi, \nabla^2\varphi].$$

4.3. A truncated system

Before considering some particular cases of the above discussion, let us recall that our initial system of differential equations (5) would actually contain some physical parameters that we have normalized to the unity up to now. In some physical situations, however, it can happen that one of these parameters is negligible and then can be put equal to zero with a good approximation [11]: this coefficient multiplies the term $[\psi, \nabla^2\varphi]$ in (5) (and in the first equation in (20)).

For this reason, it can be interesting to repeat calculations determining symmetry properties of the approximate system. The following result can be pointed out.

Proposition 2. *The truncated system*

$$\frac{\partial}{\partial t}(\psi - \nabla^2\psi) + [\varphi, \psi - \nabla^2\psi] = 0, \qquad \frac{\partial}{\partial t}(\nabla^2\varphi) + [\varphi, \nabla^2\varphi] = 0 \tag{21}$$

admits precisely the same exact symmetries as the original one (5). The same is also true if one or both of the other equations

$$[\psi, \nabla^2\varphi] = 0, \qquad [\psi, \nabla^2\psi] = 0 \tag{22}$$

are appended to the above system (therefore, even if the partial symmetry (18) is taken into consideration also within this approximation).

Observing that $[\psi, \nabla^2\varphi] = 0$ implies that $\nabla^2\varphi = B(\psi, t)$, the system (21), (22) becomes

$$\begin{aligned} &(1 - A_\psi)(\psi_t - [\psi, \varphi]) = A_t, \qquad && \nabla^2\psi = A(\psi, t), \\ &-B_\psi(\psi_t - [\psi, \varphi]) = B_t, && \nabla^2\varphi = B(\psi, t). \end{aligned}$$

Therefore, assuming, e.g., $A_\psi = 1$ forces $A_t = 0$; instead if $A_\psi \neq 1$, $B_\psi \neq 0$, but $A_t = B_t = 0$, the whole system (20) takes the very simple form

$$\psi_t = [\psi, \varphi], \qquad \nabla^2\psi = A(\psi), \qquad \nabla^2\varphi = B(\psi). \tag{23}$$

Elementary solutions of this system (and also of (5) and (20), of course) can be immediately found by simple inspection, for instance,

$$\psi = c_1 \sin(k(x - t)) + c_2 \sin(k(y - t)) + c_3, \qquad \varphi = x - y$$

where c_i, k are arbitrary constants, or also

$$\psi = 2t - \theta, \qquad \varphi = r^2$$

(where as usual $\theta = \arctan(y/x)$ and $r^2 = x^2 + y^2$), and

$$\psi = \Psi(y - t), \qquad \varphi = x$$

where Ψ is an arbitrary regular function, just to give some simple examples.

5. Equations in divergence form

It can be interesting to remark that our system (5) can be cast in the form of divergence equations. For instance, we can write (5) in the form (different but equivalent forms could also be introduced)

$$\begin{aligned} &\frac{\partial}{\partial t}(\nabla^2\psi - \psi) + \frac{\partial}{\partial x}(\psi_y\nabla^2\varphi - \varphi_y\nabla^2\psi + \psi\varphi_y) + \frac{\partial}{\partial y}(\varphi_x\nabla^2\psi - \psi_x\nabla^2\varphi - \psi\varphi_x) = 0 \\ &\frac{\partial}{\partial t}(\nabla^2\varphi) + \frac{\partial}{\partial x}(\psi_y\nabla^2\psi - \varphi_y\nabla^2\varphi) + \frac{\partial}{\partial y}(\varphi_x\nabla^2\varphi - \psi_x\nabla^2\psi) = 0 \end{aligned} \tag{24}$$

(clearly, $\psi_x = \partial\psi/\partial x$, etc). This property and some of its consequences can be summarized as follows.

Proposition 3. *The system (5) is itself a system of conserved currents of the form*

$$\frac{\partial J_0}{\partial t} + \mathrm{div}\vec{J} = 0, \qquad \frac{\partial K_0}{\partial t} + \mathrm{div}\vec{K} = 0$$

(with clear notation from (24)). The system (24), however, does not admit potential symmetries.

It is clear from the form (24) that if ψ, φ (together with $\nabla^2\psi$, $\nabla^2\varphi$) vanish rapidly enough for $|x|, |y| \to \infty$, one deduces conservation rules for the quantities J_0 and K_0

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\mathbf{R}^2}(\nabla^2\psi - \psi)\,\mathrm{d}x\,\mathrm{d}y = \frac{\mathrm{d}}{\mathrm{d}t}\iint_{\mathbf{R}^2}\Delta\varphi\,\mathrm{d}x\,\mathrm{d}y = 0$$

and for any smooth functions thereof (the same is clearly true for any domain of the plane x–y such that the flux of the vectors $\vec{J}$, $\vec{K}$ through its boundary is zero). The existence of these conservation rules has relevant consequences in plasma physics (see, e.g., [11]).

On the other hand, the divergence form (24) of our system suggests the possibility that (5) may admit *potential* symmetries. Indeed, according to an idea and a procedure suggested in [4], the initial system (5) can also be written, thanks to the form (24), in a 'potential' form, introducing a four-dimensional vector potential $P_\alpha = P_\alpha(x, y, t)$, namely

$$\begin{aligned} &\frac{\partial P_1}{\partial t} = \psi_x \nabla^2\varphi - \varphi_x(\nabla^2\psi - \psi) && \frac{\partial P_2}{\partial t} = \psi_y \nabla^2\varphi - \varphi_y(\nabla^2\psi - \psi) \\ &-\frac{\partial P_2}{\partial x} + \frac{\partial P_1}{\partial y} = \nabla^2\psi - \psi && \frac{\partial P_3}{\partial t} = \psi_x \nabla^2\psi - \varphi_x \nabla^2\varphi \\ &\frac{\partial P_4}{\partial t} = \psi_y \nabla^2\psi - \varphi_y \nabla^2\varphi && -\frac{\partial P_4}{\partial x} + \frac{\partial P_3}{\partial y} = \nabla^2\varphi \end{aligned} \tag{25}$$

where now the unknown functions are the six quantities ψ, φ and P_α.

Then one can look for standard Lie symmetries for this system, but explicit calculations show that it does not admit new symmetries which depend effectively on the potential P_α; this means that (by definition, see [4, 34]) there are no potential (nonlocal) symmetries for our problem in its original form (5). On the other hand, this conclusion is also confirmed by an extension of a result given in [34], which fixes precise necessary conditions on the order of derivatives appearing in the equations, in order that some potential symmetry can be found.

6. Conclusions

As mentioned in the introduction, the set of equations (5) are of interest for the study of the nonlinear dynamics of fluid plasma configurations and in particular for the interaction of magnetized plasma vortices and for the development of collisionless magnetic field line reconnection.

Recently in [12, 13] the onset, nonlinear evolution and saturation of magnetic reconnection instabilities have been investigated in such plasma configurations using the set of equations (5). In this framework, the explicit solutions found in the present paper using Lie point symmetries have a direct physical interpretation, e.g., in the case of the solutions (8) as plasma configurations forced from the boundaries. Further investigation of the physical properties of these configurations and the effect of the forcing on the onset and evolution of the reconnection instabilities will be the subject of a forthcoming paper aimed at investigating new mechanisms of control of the reconnection instabilities.

Appendix

The system of equations investigated in this paper has been derived in [35] and is used in the plasma physics literature [10–16] in order to describe the 'low-frequency' dynamics of a plasma embedded in a strong, almost uniform magnetic field in regimes where kinetic processes, as opposed to fluid processes, can be neglected. This system of equations is derived from the so-called 'two-fluid' description of the plasma dynamics which generalizes the well-known magnetohydrodynamic (MHD) equations by accounting for effects, such as electron inertia and electron compressibility along field lines, that are not included in the standard MHD description. These electron effects are outside the scope of the single-fluid approximation that is at the basis of the MHD description, but need to be included in order to account for phenomena, such as magnetic field line reconnection in dissipationless plasma regimes, where resistive and viscous terms are ineffective. These regimes are of interest for present day magnetic fusion experiments and for space plasmas. When such non-dissipative electron effects are disregarded, the standard 'ideal' MHD equations are recovered from the two-fluid approach.

The two-fluid system of equations is then specialized, as in MHD where it leads to the so-called reduced MHD equations [36, 37], to the case where the plasma pressure is much smaller that the magnetic pressure because of the presence of a strong, uniform magnetic field component along a direction that we take to coincide with the z-axis. This reduction is possible since the presence of this field makes the low-frequency plasma behaviour essentially two dimensional and restricted to the x–y plane.

Thus, in deriving the system of equations investigated in this paper, we refer to a plasma configuration that is homogeneous in the z-direction and write the electric and magnetic fields in the plasma as

$$\mathbf{E} = -\nabla\varphi - \frac{1}{c}\frac{\partial\psi}{\partial t}\mathbf{e}_z, \qquad \mathbf{B} = B_0\mathbf{e}_z + \nabla\psi \times \mathbf{e}_z. \tag{A.1}$$

Here φ and ψ are the electrostatic and the vector potential, respectively, B_0 is a constant, and $\mathbf{e}_z$ is the unit vector in the z-direction. The electron momentum balance along magnetic field lines and the continuity equations give

$$m_e n \frac{\mathrm{d}v_z}{\mathrm{d}t} = -enE_\parallel - \nabla_\parallel p, \tag{A.2}$$

and

$$\frac{\mathrm{d}n}{\mathrm{d}t} + \nabla_\parallel(nv_z) = 0, \tag{A.3}$$

where n and p are the electron density and pressure, v_z is the parallel electron velocity, $\mathrm{d}/\mathrm{d}t = \partial/\partial t + \mathbf{v}_E \cdot \nabla$. Here $\mathbf{v}_E = (c/B_0)[\mathbf{e}_z \times \nabla\varphi]$ is the $\mathbf{E} \times \mathbf{B}$ drift velocity, and $\nabla_\parallel = \partial/\partial z + (1/B_0)[\nabla\psi \times \mathbf{e}_z] \cdot \nabla$ is the derivative along the magnetic field. Then, following the procedure detailed in [35], we write Ampère's law as

$$\nabla^2\psi = 4\pi e n_0 v_z/c, \tag{A.4}$$

where n_0 is a reference density value. As in the MHD description, we apply the quasineutrality condition $n \approx n_i$, where n_i is the ion density. The ion density is obtained by solving the perpendicular ion momentum balance equation for the perpendicular fluid ion velocity and by inserting it in the ion continuity equation. Then we have

$$\frac{\mathrm{d}n}{\mathrm{d}t} - \nabla\left(\frac{nc}{\Omega_i B_0}\frac{\mathrm{d}\nabla\varphi}{\mathrm{d}t}\right) = 0, \tag{A.5}$$

where Ω_i is the ion cyclotron frequency and the term inside the parentheses arises from the so-called ion polarization drift. Then, using the Poisson brackets defined below equation (5) and adopting suitable normalizations, from equations (A.2)–(A.5) we obtain

$$\frac{\partial F}{\partial t} + [\varphi, F] + \rho_s^2[\psi, U] = 0, \tag{A.6}$$

$$\frac{\partial U}{\partial t} + [\varphi, U] - [J, \psi] = 0, \tag{A.7}$$

where $F \equiv \psi + d_e^2 J$, $J = -\nabla^2\psi$ is the plasma current density along z and $U = \nabla^2\varphi$ is the fluid vorticity. The electron inertia skin depth d_e and the ion sound gyroradius ρ_s are defined by

$$d_e^2 = c^2 m_e/(4\pi n_0 e^2) \quad \text{and} \quad \rho_s^2 = (T_e/m_i)/\Omega_i^2. \tag{A.8}$$

Then, by summing and subtracting equations (A.6), (A.7), we obtain the system

$$\frac{\partial G_\pm}{\partial t} + [\varphi_\pm, G_\pm] = 0, \tag{A.9}$$

where the fields $G_\pm$ and the streaming potentials $\varphi_\pm$ are defined by

$$G_\pm = F \pm d_e \rho_s U \qquad \text{and} \qquad \varphi_\pm = \varphi(\rho_s/d_e)\psi. \tag{A.10}$$

The system of equations (5) is then obtained by rescaling lengths and times and by setting the dimensionless ratio $\rho_s/d_e = 1$ for the sake of simplicity. It is worth noting that each of the two equations (A.9) has the same algebraic structure of the incompressible two-dimensional Euler equation for the fluid vorticity and that equation (A.7) reduces to the two-dimensional Euler equation in the hydrodynamical limit $\psi \equiv 0$.

Finally, we note that equations (5) obey boundary conditions that are determined by values (at the boundaries of the domain where the plasma is enclosed) of the electric and magnetic fields given by equation (A.1). These boundary conditions may include plasma fluxes through the boundaries. Referring for an example to the solutions defined by equation (8) we see that they correspond to a plasma velocity pattern controlled by the boundary conditions that determine the arbitrary functions $A(t)$ and $B(t)$. If we imagine that these plasma configurations are confined between two plates at $z = z_1, z_2$, the required boundary conditions correspond to imposing the electric field at these plates: the tangential component is proportional to $\nabla\varphi$ and determines the velocity pattern, while the normal component is proportional to $\partial\psi/\partial t$ and determines the magnetic terms. Clearly, the boundary conditions on ψ are related to those imposed on φ. These boundary conditions at the top and bottom plates determine the value of the electric and magnetic fields imposed at the boundary of the plasma domain in the x–y plane.

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